

Perturbation of the Kerr Metric

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Abstract

A new Kerr-like metric with quadrupole moment is obtained by means of perturbing the Kerr spacetime. By comparison with the exterior Hartle-Thorne metric, it is showed that it could be matched to an interior solution. This metric may represent the spacetime of an astrophysical object.

1 Introduction

In 1963, R. P. Kerr [17] proposed a metric that describes a massive rotating object. Since then, a huge amount of papers about the structure and astrophysical applications of this spacetime appeared. Now, it is widely believed that this metric does not represent the spacetime of an astrophysical rotating object. This is because the Kerr metric cannot be matched to a realistic interior metric [2].

Other multipole and rotating solutions to the Einstein field equations (EFE) were obtained by Castejón *et al.* (1990) [4], Manko & Novikov (1992) [19], Manko *et al.* (2000) [20], Pachon *et al.* (2006) [21], and Quevedo (1986) [22], Quevedo (1989) [23], Quevedo & Mashhoon [24], Quevedo (2011) [25]. In the four first articles, they used the Ernst formalism [7], while in the four last ones, the solutions were obtained with the help of the Hoenselaers-Kinnersley-Xanthopoulos (HKX) transformations [16]. These authors obtain new metrics from a given seed metric. These formalisms allow to include other desirable characteristics (rotation, multipole moments, magnetic dipole, etc.) to a given seed metrics.

In Nature, it is expected that astrophysical objects are rotating and slightly deformed. The aim of this article is to derive an appropriate analytical tractable metric for calculations in which the quadrupole moment can be

treated as perturbation, but for arbitrary angular momentum. Moreover, this metric should be useful to tackle astrophysical problems, for instance, accretion disk in compact stellar objects [9, 14], relativistic magnetohydrodynamic jet formation [8], astrometry [26, 12] and gravitational lensing [10]. Furthermore, software related with applications of the Kerr metric can be easily modified in order to include the quadrupole moment [6, 27, 11].

This paper is organized as follows. In section 2, we give a succinct explanation of the Kerr metric. The weak limit of the Erez-Rosen metric is presented in section 3. In section 4, the Lewis metric is presented. The perturbation method is discussed in section 5. The application of this method leads to a new solution to the EFE with quadrupole moment and rotation. It is checked by means of the REDUCE software [15] that the resulting metric is solution of the EFE. In section 6, we compare our solution with the exterior Hartle-Thorne metric in order to assure that our metric has astrophysical meaning. Forthcoming works with this metric are discussed in section 7.

2 The Kerr Metric

The Kerr metric represents the spacetime of a non-deformed massive rotating object. The Kerr metric is given by [17, 3]

$$ds^2 = \frac{\Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 - \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2, \quad (1)$$

where $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2 \cos^2 \theta$. M and a represent the mass and the rotation parameter, respectively. The angular momentum of the object is $J = Ma$.

3 The Erez-Rosen metric

The Erez-Rosen metric [3, 28, 29, 30] represents a body with quadrupole moment. The principal axis of the quadrupole moment is chosen along the spin axis, so that gravitational radiation can be ignored. Here, we write down an approximate expression for this metric obtained by doing Taylor series [12]

$$ds^2 = \left(1 - \frac{2M}{r}\right) e^{-2\chi} dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} e^{2\chi} dr^2 - r^2 e^{2\chi} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2)$$

where $d\Sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2$, and

$$\chi = \frac{2}{15} q \frac{M^3}{r^3} P_2(\cos \theta). \quad (3)$$

The quadrupole parameter is given by $q = 15GQ/(2c^2 M^3)$, with Q representing the quadrupole moment. This metric is valid up to the order $O(qM^4, q^2)$.

4 The Lewis Metrics

The Lewis metric is given by [18, 3]

$$ds^2 = V dt^2 - 2W dt d\phi - e^\mu d\rho^2 - e^\nu dz^2 - Z d\phi^2 \quad (4)$$

where we have chosen the canonical coordinates $x^1 = \rho$ and $x^2 = z$, V , W , Z , μ and ν are functions of ρ and z ($\rho^2 = VZ + W^2$). Choosing $\mu = \nu$ and performing the following changes of potentials

$$V = f, \quad W = \omega f, \quad Z = \frac{\rho^2}{f} - \omega^2 f \quad \text{and} \quad e^\mu = \frac{e^\gamma}{f},$$

we get the Papapetrou metric

$$ds^2 = f(dt - \omega d\phi)^2 - \frac{e^\gamma}{f} [d\rho^2 + dz^2] - \frac{\rho^2}{f} d\phi^2. \quad (5)$$

5 Perturbing the Kerr Metric

To include a small quadrupole moment into the Kerr metric we will modify the Lewis-Papapetrou metric (5). First of all, we choose expressions for the canonical coordinates ρ and z . For the Kerr metric [17], one particular choice is [3, 5]

$$\rho = \sqrt{\Delta} \sin \theta \quad \text{and} \quad z = (r - M) \cos \theta \quad (6)$$

where $\Delta = r^2 - 2Mr + a^2$.

From (6) we get

$$d\rho^2 + dz^2 = [(r - M)^2 \sin^2 \theta + \Delta \cos^2 \theta] \left(\frac{dr^2}{\Delta} + d\theta^2 \right). \quad (7)$$

If we choose

$$e^\mu = \tilde{\rho}^2 [(r - M)^2 \sin^2 \theta + \Delta \cos^2 \theta]^{-1},$$

the term (7) becomes

$$e^\mu [d\rho^2 + dz^2] = \tilde{\rho}^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right),$$

where $\tilde{\rho}^2 = r^2 + a^2 \cos^2 \theta$.

From (5), we propose the following metric

$$ds^2 = \mathcal{V} dt^2 - 2\mathcal{W} dt d\phi - \mathcal{X} dr^2 - \mathcal{Y} d\theta^2 - \mathcal{Z} d\phi^2, \quad (8)$$

where

$$\begin{aligned} \mathcal{V} &= V e^{-2\psi} \\ \mathcal{W} &= W \\ \mathcal{X} &= X e^{2\psi} \\ \mathcal{Y} &= Y e^{2\psi} \\ \mathcal{Z} &= Z e^{2\psi}, \end{aligned} \quad (9)$$

where the potentials V, W, X, Y, Z , and ψ depend on $x^1 = r$ and $x^2 = \theta$.

Now, let us choose

$$\begin{aligned} V &= f = \frac{1}{\tilde{\rho}^2} [\Delta - a^2 \sin^2 \theta] \\ W &= \frac{a}{\tilde{\rho}^2} [\Delta - (r^2 + a^2)] \sin^2 \theta = -\frac{2Jr}{\tilde{\rho}^2} \sin^2 \theta \\ X &= \frac{\tilde{\rho}^2}{\Delta} \\ Y &= \tilde{\rho}^2 \\ Z &= \frac{\sin^2 \theta}{\tilde{\rho}^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta]. \end{aligned} \quad (10)$$

The only potential we have to find is ψ . In order to obtain this potential, the EFE must be solved

$$G_{ij} = R_{ij} - \frac{R}{2}g_{ij} = 0 \quad (11)$$

where R_{ij} ($i, j = 0, 1, 2, 3$) are the Ricci tensor components and R is the curvature scalar. The Ricci tensor components and the curvature scalar R for this metric can be found in the Appendix.

In our calculations, we consider the potential ψ as perturbation, *i.e.* one neglects terms of the form

$$\left(\frac{\partial\psi}{\partial r}\right)^2 = \left(\frac{\partial\psi}{\partial\theta}\right)^2 = \frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial\theta} \sim 0.$$

Terms containing factors of the form

$$a\frac{\partial\psi}{\partial x^i} = m\frac{\partial\psi}{\partial x^i} \sim 0 \quad (i = 1, 2)$$

are also neglected. Substituting the known potentials (V, W, X, Y, Z) into the expressions for the Ricci tensor and the curvature scalar (see Appendix), it results only one equation for ψ that we have to solved:

$$\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0 \quad (12)$$

The solution for this equation is

$$\psi = \frac{\mathcal{K}}{r^3}P_2(\cos\theta), \quad (13)$$

where \mathcal{K} is a constant. To determine this constant, we compare the weak limit of the metric (8) with the Erez-Rosen metric (2), *i.e.* $\psi = \chi$. The result is $\mathcal{K} = 2qM^3/15$.

Then, the new modified Kerr metric containing quadrupole moment is

$$\begin{aligned}
ds^2 &= \frac{e^{-2\chi}}{\rho^2} [\Delta - a^2 \sin^2 \theta] dt^2 + \frac{4Jr}{\rho^2} \sin^2 \theta dt d\phi - \frac{\rho^2 e^{2\chi}}{\Delta} dr^2 - \rho^2 e^{2\chi} d\theta^2 \\
&- \frac{e^{2\chi} \sin^2 \theta}{\rho^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] d\phi^2 \\
&= \frac{\Delta}{\rho^2} [e^{-\chi} dt - a e^{\chi} \sin^2 \theta d\phi]^2 - \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) e^{\chi} d\phi - a e^{-\chi} dt]^2 \\
&- e^{2\chi} \left(\frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \right), \tag{14}
\end{aligned}$$

where the tilde over the ρ is dropped.

We verified that the metric (14) is indeed a solution of the EFE using REDUCE [15] up to the order $O(qM^4, q^2)$.

6 Comparison with the Exterior Hartle-Thorne Metric

In order to establish whether the metric (14) does really represent the gravitational field of an astrophysical object, we should show that it is possible to construct an interior solution, which can appropriately be matched with the exterior solution. For this purpose, Boshkayev *et al.* [2] and Frutos-Alfaro *et al.* [12] employed the exterior Hartle-Thorne metric [13, 1]

$$\begin{aligned}
ds^2 &= \left(1 - \frac{2\mathcal{M}}{r} + \frac{2\mathcal{Q}\mathcal{M}^3}{r^3} P_2(\cos \theta) \right) dt^2 \\
&- \left(1 + \frac{2\mathcal{M}}{r} + \frac{4\mathcal{M}^2}{r^2} - \frac{2\mathcal{Q}\mathcal{M}^3}{r^3} P_2(\cos \theta) \right) dr^2 \\
&- r^2 \left(1 - \frac{2\mathcal{Q}\mathcal{M}^3}{r^3} P_2(\cos \theta) \right) d\Sigma^2 + \frac{4\mathcal{J}}{r} \sin^2 \theta dt d\phi, \tag{15}
\end{aligned}$$

where \mathcal{M} , \mathcal{J} , and \mathcal{Q} are related with the total mass, angular momentum, and mass quadrupole moment of the rotating object, respectively.

The spacetime (14) has the same weak limit as the metric obtained by Frutos *et al.* [12]. A comparison of the exterior Hartle-Thorne metric [13] with the weak limit of the metric (14) shows that upon defining

$$\mathcal{M} = M, \quad \mathcal{J} = J, \quad 2Q\mathcal{M}^3 = -\frac{4}{15}qM^3, \quad (16)$$

both metrics coincide up to the order $O(M^3, a^2, qM^4, q^2)$. Hence, the metric (14) may be used to represent a compact astrophysical object.

7 Conclusions

The new Kerr metric with quadrupole moment was obtained by solving the EFE approximately. It may represent the spacetime of a rotating and slightly deformed astrophysical object. This is possible, because it could be matched to an interior solution. We showed it by comparison of our metric with the exterior Hartle-Thorne metric. Moreover, the inclusion of the quadrupole moment in the Kerr metric does it more suitable for astrophysical calculations than the Kerr metric alone. There are a large variety of applications which can be tackled with this new metric. Amongst the applications for this metric are astrometry, gravitational lensing, relativistic magnetohydrodynamic jet formation, and accretion disks in compact stellar objects. Furthermore, the existing software with applications of the Kerr metric can be easily modified to include the quadrupole moment.

A Appendix

$$\begin{aligned}
R_{00} &= \frac{e^{-2\psi}}{4\rho^2 X^2 Y^2} \left(-4\rho^2 V X^2 Y \frac{\partial^2 \psi}{\partial \theta^2} + 8V W^2 X^2 Y \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right. \\
&\quad - 2\rho^2 V X Y \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} + 2V X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial \rho^2}{\partial \theta} - 4\rho^2 X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial V}{\partial \theta} \\
&\quad - 4W^2 X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial V}{\partial \theta} + 2\rho^2 V X^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} - 4V^2 X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
&\quad - 4\rho^2 V X Y^2 \frac{\partial^2 \psi}{\partial r^2} + 8V W^2 X Y^2 \left(\frac{\partial \psi}{\partial r} \right)^2 + 2\rho^2 V Y^2 \frac{\partial \psi}{\partial r} \frac{\partial X}{\partial r} \\
&\quad + 2V X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial \rho^2}{\partial r} - 4\rho^2 X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial V}{\partial r} - 4W^2 X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial V}{\partial r} \\
&\quad - 2\rho^2 V X Y \frac{\partial \psi}{\partial r} \frac{\partial Y}{\partial r} - 4V^2 X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial Z}{\partial r} + \rho^2 X Y \frac{\partial X}{\partial \theta} \frac{\partial V}{\partial \theta} \\
&\quad - \rho^2 Y^2 \frac{\partial X}{\partial r} \frac{\partial V}{\partial r} - X^2 Y \frac{\partial \rho^2}{\partial \theta} \frac{\partial V}{\partial \theta} - X Y^2 \frac{\partial \rho^2}{\partial r} \frac{\partial V}{\partial r} \\
&\quad + 2\rho^2 X^2 Y \frac{\partial^2 V}{\partial \theta^2} - \rho^2 X^2 \frac{\partial V}{\partial \theta} \frac{\partial Y}{\partial \theta} + 2V X^2 Y \frac{\partial V}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
&\quad + 2\rho^2 X Y^2 \frac{\partial^2 V}{\partial r^2} + \rho^2 X Y \frac{\partial V}{\partial r} \frac{\partial Y}{\partial r} + 2V X Y^2 \frac{\partial V}{\partial r} \frac{\partial Z}{\partial r} \\
&\quad \left. + 2V X^2 Y \left(\frac{\partial W}{\partial \theta} \right)^2 + 2V X Y^2 \left(\frac{\partial W}{\partial r} \right)^2 \right) \\
R_{01} &= 0 \\
R_{02} &= 0
\end{aligned}$$

$$\begin{aligned}
R_{03} = & \frac{e^{-2\psi}}{4\rho^2 X^2 Y^2} \left(8\rho^2 W X^2 Y \left(\frac{\partial\psi}{\partial\theta} \right)^2 - 8W^3 X^2 Y \left(\frac{\partial\psi}{\partial\theta} \right)^2 \right. \\
& - 4W X^2 Y \frac{\partial\psi}{\partial\theta} \frac{\partial\rho^2}{\partial\theta} + 8W^2 X^2 Y \frac{\partial\psi}{\partial\theta} \frac{\partial W}{\partial\theta} + 8V W X^2 Y \frac{\partial\psi}{\partial\theta} \frac{\partial Z}{\partial\theta} \\
& + 8\rho^2 W X Y^2 \left(\frac{\partial\psi}{\partial r} \right)^2 - 8W^3 X Y^2 \left(\frac{\partial\psi}{\partial r} \right)^2 - 4W X Y^2 \frac{\partial\psi}{\partial r} \frac{\partial\rho^2}{\partial r} \\
& + 8W^2 X Y^2 \frac{\partial\psi}{\partial r} \frac{\partial W}{\partial r} + 8V W X Y^2 \frac{\partial\psi}{\partial r} \frac{\partial Z}{\partial r} - \rho^2 X Y \frac{\partial X}{\partial\theta} \frac{\partial W}{\partial\theta} \\
& + \rho^2 Y^2 \frac{\partial X}{\partial r} \frac{\partial W}{\partial r} + X^2 Y \frac{\partial\rho^2}{\partial\theta} \frac{\partial W}{\partial\theta} + X Y^2 \frac{\partial\rho^2}{\partial r} \frac{\partial W}{\partial r} \\
& - 2W X^2 Y \frac{\partial V}{\partial\theta} \frac{\partial Z}{\partial\theta} - 2W X Y^2 \frac{\partial V}{\partial r} \frac{\partial Z}{\partial r} - 2\rho^2 X^2 Y \frac{\partial^2 W}{\partial\theta^2} \\
& - 2W X^2 Y \left(\frac{\partial W}{\partial\theta} \right)^2 + \rho^2 X^2 \frac{\partial W}{\partial\theta} \frac{\partial Y}{\partial\theta} - 2\rho^2 X Y^2 \frac{\partial^2 W}{\partial r^2} \\
& \left. - 2W X Y^2 \left(\frac{\partial W}{\partial r} \right)^2 - \rho^2 X Y \frac{\partial W}{\partial r} \frac{\partial Y}{\partial r} \right)
\end{aligned}$$

$$\begin{aligned}
R_{11} = & \frac{1}{4\rho^4 XY^2} \left(-4\rho^4 X^2 Y \frac{\partial^2 \psi}{\partial \theta^2} - 2\rho^4 XY \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} \right. \\
& - 2\rho^2 X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial \rho^2}{\partial \theta} + 2\rho^4 X^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} - 4\rho^4 XY^2 \frac{\partial^2 \psi}{\partial r^2} \\
& - 8\rho^4 XY^2 \frac{\partial \psi^2}{\partial r} + 8\rho^2 W^2 XY^2 \left(\frac{\partial \psi}{\partial r} \right)^2 + 2\rho^4 Y^2 \frac{\partial \psi}{\partial r} \frac{\partial X}{\partial r} \\
& + 6\rho^2 XY^2 \frac{\partial \psi}{\partial r} \frac{\partial \rho^2}{\partial r} - 8\rho^2 W XY^2 \frac{\partial \psi}{\partial r} \frac{\partial W}{\partial r} - 2\rho^4 XY \frac{\partial \psi}{\partial r} \frac{\partial Y}{\partial r} \\
& - 8\rho^2 V XY^2 \frac{\partial \psi}{\partial r} \frac{\partial Z}{\partial r} - 2\rho^4 XY \frac{\partial^2 X}{\partial \theta^2} + \rho^4 Y \left(\frac{\partial X}{\partial \theta} \right)^2 \\
& - \rho^2 XY \frac{\partial X}{\partial \theta} \frac{\partial \rho^2}{\partial \theta} + \rho^4 X \frac{\partial X}{\partial \theta} \frac{\partial Y}{\partial \theta} + \rho^2 Y^2 \frac{\partial X}{\partial r} \frac{\partial \rho^2}{\partial r} \\
& + \rho^4 Y \frac{\partial X}{\partial r} \frac{\partial Y}{\partial r} - 2\rho^2 XY^2 \frac{\partial^2 \rho^2}{\partial r^2} + XY^2 \left(\frac{\partial \rho^2}{\partial r} \right)^2 \\
& + 2V XY^2 \frac{\partial \rho^2}{\partial r} \frac{\partial Z}{\partial r} + 2W^2 XY^2 \frac{\partial V}{\partial r} \frac{\partial Z}{\partial r} + 2\rho^2 XY^2 \left(\frac{\partial W}{\partial r} \right)^2 \\
& - 4VW XY^2 \frac{\partial W}{\partial r} \frac{\partial Z}{\partial r} - 2\rho^4 XY \frac{\partial^2 Y}{\partial r^2} + \rho^4 X \left(\frac{\partial Y}{\partial r} \right)^2 \\
& \left. - 2V^2 XY^2 \left(\frac{\partial Z}{\partial r} \right)^2 \right)
\end{aligned}$$

$$\begin{aligned}
R_{12} &= \frac{1}{4\rho^4 XY} \left(-8\rho^4 XY \frac{\partial\psi}{\partial\theta} \frac{\partial\psi}{\partial r} + 8\rho^2 W^2 XY \frac{\partial\psi}{\partial\theta} \frac{\partial\psi}{\partial r} \right. \\
&+ 4\rho^2 XY \frac{\partial\psi}{\partial\theta} \frac{\partial\rho^2}{\partial r} - 4\rho^2 W XY \frac{\partial\psi}{\partial\theta} \frac{\partial W}{\partial r} - 4\rho^2 V XY \frac{\partial\psi}{\partial\theta} \frac{\partial Z}{\partial r} \\
&+ 4\rho^2 XY \frac{\partial\psi}{\partial r} \frac{\partial\rho^2}{\partial\theta} - 4\rho^2 W XY \frac{\partial\psi}{\partial r} \frac{\partial W}{\partial\theta} - 4\rho^2 V XY \frac{\partial\psi}{\partial r} \frac{\partial Z}{\partial\theta} \\
&+ \rho^2 Y \frac{\partial X}{\partial\theta} \frac{\partial\rho^2}{\partial r} - 2\rho^2 XY \frac{\partial^2\rho^2}{\partial\theta\partial r} + W^2 XY \frac{\partial^2\rho^2}{\partial\theta\partial r} \\
&+ XY \frac{\partial\rho^2}{\partial\theta} \frac{\partial\rho^2}{\partial r} + \rho^2 X \frac{\partial\rho^2}{\partial\theta} \frac{\partial Y}{\partial r} + V XY \frac{\partial\rho^2}{\partial\theta} \frac{\partial Z}{\partial r} \\
&+ V XY \frac{\partial\rho^2}{\partial r} \frac{\partial Z}{\partial\theta} - W^2 XY Z \frac{\partial^2 V}{\partial\theta\partial r} - 2W^3 XY \frac{\partial^2 W}{\partial\theta\partial r} \\
&+ 2\rho^2 XY \frac{\partial W}{\partial\theta} \frac{\partial W}{\partial r} - 2W^2 XY \frac{\partial W}{\partial\theta} \frac{\partial W}{\partial r} - 2VW XY \frac{\partial W}{\partial\theta} \frac{\partial Z}{\partial r} \\
&- 2VW XY \frac{\partial W}{\partial r} \frac{\partial Z}{\partial\theta} - VW^2 XY \frac{\partial^2 Z}{\partial\theta\partial r} - 2V^2 XY \frac{\partial Z}{\partial\theta} \frac{\partial Z}{\partial r} \Big) \\
R_{13} &= 0
\end{aligned}$$

$$\begin{aligned}
R_{22} &= \frac{1}{4\rho^4 X^2 Y} \left(-4\rho^4 X^2 Y \frac{\partial^2 \psi}{\partial \theta^2} - 8\rho^4 X^2 Y \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right. \\
&+ 8\rho^2 W^2 X^2 Y \left(\frac{\partial \psi}{\partial \theta} \right)^2 - 2\rho^4 X Y \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} + 6\rho^2 X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial \rho^2}{\partial \theta} \\
&- 8\rho^2 W X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial W}{\partial \theta} + 2\rho^4 X^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} - 8\rho^2 V X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
&- 4\rho^4 X Y^2 \frac{\partial^2 \psi}{\partial r^2} + 2\rho^4 Y^2 \frac{\partial \psi}{\partial r} \frac{\partial X}{\partial r} - 2\rho^2 X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial \rho^2}{\partial r} \\
&- 2\rho^4 X Y \frac{\partial \psi}{\partial r} \frac{\partial Y}{\partial r} - 2\rho^4 X Y \frac{\partial^2 X}{\partial \theta^2} + \rho^4 Y \left(\frac{\partial X}{\partial \theta} \right)^2 \\
&+ \rho^4 X \frac{\partial X}{\partial \theta} \frac{\partial Y}{\partial \theta} + \rho^4 Y \frac{\partial X}{\partial r} \frac{\partial Y}{\partial r} - 2\rho^2 X^2 Y \frac{\partial^2 \rho^2}{\partial \theta^2} \\
&+ X^2 Y \left(\frac{\partial \rho^2}{\partial \theta} \right)^2 + \rho^2 X^2 \frac{\partial \rho^2}{\partial \theta} \frac{\partial Y}{\partial \theta} + 2V X^2 Y \frac{\partial \rho^2}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
&- \rho^2 X Y \frac{\partial \rho^2}{\partial r} \frac{\partial Y}{\partial r} + 2W^2 X^2 Y \frac{\partial V}{\partial \theta} \frac{\partial Z}{\partial \theta} + 2\rho^2 X^2 Y \left(\frac{\partial W}{\partial \theta} \right)^2 \\
&- 4VW X^2 Y \frac{\partial W}{\partial \theta} \frac{\partial Z}{\partial \theta} - 2\rho^4 X Y \frac{\partial^2 Y}{\partial r^2} + \rho^4 X \left(\frac{\partial Y}{\partial r} \right)^2 \\
&\left. - 2V^2 X^2 Y \left(\frac{\partial Z}{\partial \theta} \right)^2 \right) \\
R_{23} &= 0
\end{aligned}$$

$$\begin{aligned}
R_{33} = & \frac{1}{4\rho^2 X^2 Y^2} \left(-4\rho^2 X^2 Y Z \frac{\partial^2 \psi}{\partial \theta^2} - 8W^2 X^2 Y Z \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right. \\
& - 2\rho^2 X Y Z \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} - 2Y X^2 Z \frac{\partial \psi}{\partial \theta} \frac{\partial \rho^2}{\partial \theta} + 8W X^2 Y Z \frac{\partial \psi}{\partial \theta} \frac{\partial W}{\partial \theta} \\
& + 2\rho^2 X^2 Z \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} - 8W^2 X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial \theta} - 4\rho^2 X Y^2 Z \frac{\partial^2 \psi}{\partial r^2} \\
& - 8W^2 X Y^2 Z \left(\frac{\partial \psi}{\partial r} \right)^2 + 2\rho^2 Y^2 Z \frac{\partial \psi}{\partial r} \frac{\partial X}{\partial r} - 2X Y^2 Z \frac{\partial \psi}{\partial r} \frac{\partial \rho^2}{\partial r} \\
& + 8W X Y^2 Z \frac{\partial \psi}{\partial r} \frac{\partial W}{\partial r} - 2\rho^2 X Y Z \frac{\partial \psi}{\partial r} \frac{\partial Y}{\partial r} - 8W^2 X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial Z}{\partial r} \\
& - \rho^2 X Y \frac{\partial X}{\partial \theta} \frac{\partial Z}{\partial \theta} + \rho^2 Y^2 \frac{\partial X}{\partial r} \frac{\partial Z}{\partial r} - X^2 Y \frac{\partial \rho^2}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
& - X Y^2 \frac{\partial \rho^2}{\partial r} \frac{\partial Z}{\partial r} - 2X^2 Y Z \left(\frac{\partial W}{\partial \theta} \right)^2 + 4W X^2 Y \frac{\partial W}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
& - 2X Y^2 Z \left(\frac{\partial W}{\partial r} \right)^2 + 4W X Y^2 \frac{\partial W}{\partial r} \frac{\partial Z}{\partial r} + \rho^2 X^2 \frac{\partial Y}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
& - \rho^2 X Y \frac{\partial Y}{\partial r} \frac{\partial Z}{\partial r} - 2\rho^2 X^2 Y \frac{\partial^2 Z}{\partial \theta^2} + 2V X^2 Y \left(\frac{\partial Z}{\partial \theta} \right)^2 \\
& \left. - 2\rho^2 X Y^2 \frac{\partial^2 Z}{\partial r^2} + 2V X Y^2 \left(\frac{\partial Z}{\partial r} \right)^2 \right)
\end{aligned}$$

Calculation of the scalar curvature

$$\begin{aligned}
R = & \frac{e^{-2\psi}}{2\rho^4 X^2 Y^2} \left(4\rho^4 X^2 Y \frac{\partial^2 \psi}{\partial \theta^2} + 4\rho^4 X^2 Y \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right. \\
& - 4\rho^2 W^2 X^2 Y \left(\frac{\partial \psi}{\partial \theta} \right)^2 + 2\rho^4 X Y \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \theta} - 2\rho^2 X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial \rho^2}{\partial \theta} \\
& + 4\rho^2 W X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial W}{\partial \theta} - 2\rho^4 X^2 \frac{\partial \psi}{\partial \theta} \frac{\partial Y}{\partial \theta} + 4\rho^2 V X^2 Y \frac{\partial \psi}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
& + 4\rho^4 X Y^2 \frac{\partial^2 \psi}{\partial r^2} + 4\rho^4 X Y^2 \left(\frac{\partial \psi}{\partial r} \right)^2 - 4\rho^2 W^2 X Y^2 \left(\frac{\partial \psi}{\partial r} \right)^2 \\
& - 2\rho^4 Y^2 \frac{\partial \psi}{\partial r} \frac{\partial X}{\partial r} - 2\rho^2 X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial \rho^2}{\partial r} + 4\rho^2 W X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial W}{\partial r} \\
& + 2\rho^4 X Y \frac{\partial \psi}{\partial r} \frac{\partial Y}{\partial r} + 4\rho^2 V X Y^2 \frac{\partial \psi}{\partial r} \frac{\partial Z}{\partial r} + 2\rho^4 X Y \frac{\partial^2 X}{\partial \theta^2} \\
& - \rho^4 Y \left(\frac{\partial X}{\partial \theta} \right)^2 + \rho^2 X Y \frac{\partial X}{\partial \theta} \frac{\partial \rho^2}{\partial \theta} - \rho^4 X \frac{\partial X}{\partial \theta} \frac{\partial Y}{\partial \theta} \\
& - \rho^2 Y^2 \frac{\partial X}{\partial r} \frac{\partial \rho^2}{\partial r} - \rho^4 Y \frac{\partial X}{\partial r} \frac{\partial Y}{\partial r} + 2\rho^2 X^2 Y \frac{\partial^2 \rho^2}{\partial \theta^2} \\
& - X^2 Y \left(\frac{\partial \rho^2}{\partial \theta} \right)^2 - \rho^2 X^2 \frac{\partial \rho^2}{\partial \theta} \frac{\partial Y}{\partial \theta} + 2\rho^2 X Y^2 \frac{\partial^2 \rho^2}{\partial r^2} \\
& - X Y^2 \left(\frac{\partial \rho^2}{\partial r} \right)^2 + \rho^2 X Y \frac{\partial \rho^2}{\partial r} \frac{\partial Y}{\partial r} - \rho^2 X^2 Y \frac{\partial V}{\partial \theta} \frac{\partial Z}{\partial \theta} \\
& - \rho^2 X Y^2 \frac{\partial V}{\partial r} \frac{\partial Z}{\partial r} - \rho^2 X^2 Y \left(\frac{\partial W}{\partial \theta} \right)^2 - \rho^2 X Y^2 \left(\frac{\partial W}{\partial r} \right)^2 \\
& \left. + 2\rho^4 X Y \frac{\partial^2 Y}{\partial r^2} - \rho^4 X \left(\frac{\partial Y}{\partial r} \right)^2 \right)
\end{aligned}$$

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